Embedded Software Laboratory

RWTH AACHEN UNIVERSITY

Approximate Quantifier Elimination for Propositional Boolean Formulae

Jörg Brauer **Andy King**

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Motivation

- Quantifier elimination on Boolean formulae in
  - Unbounded symbolic model checking, predicate abstraction, dependency analysis, transfer function synthesis, information flow analysis, ranking function synthesis, etc.
- Computationally expensive operation
  - Model enumeration using SAT possible
  - Still potentially too expensive
  - Especially when result should be in CNF

Approach

- To compute $\exists x_1, \ldots, x_n : \varphi$ in CNF, you classically eliminate the x_i one after another
- Only final result is free of $x_1, \ldots, x_n$
- We compute C_i such that $\exists x_1, \ldots, x_n : \varphi \models C_i$
  – Then C_i over-approximates $\exists x_1, \ldots, x_n : \varphi$
- Refine over-approximation as $\exists x_1, \ldots, x_n : \varphi \models C_i \wedge C_j$
- The C clauses derived from prime implicants

Dual-Rail Encoding for Implicants

- Consider

$$\varphi = (\neg x \lor z) \land (y \lor z) \land (\neg x \lor \neg w \lor \neg z) \land (w \lor \neg z)$$

- Goal: eliminate z from φ such that $\exists z : \varphi$ in CNF
- Dual-rail encoding
  - Introduce fresh variables
  - Replace positive and negative literals

$$\tau(\varphi) = \begin{cases} (x^- \lor z) \land (y^+ \lor z) \land (x^- \lor w^- \lor \neg z) \land (w^+ \lor \neg z) \land \\ (\neg w^+ \lor \neg w^-) \land (\neg x^+ \lor \neg x^-) \land (\neg y^+ \lor \neg y^-) \end{cases}$$

Dual-Rail Encoding for Implicants

- Passing $\tau(\varphi)$ to SAT solver gives a model

$$\mathcal{M} = \left\{ \begin{array}{llllllllllll} w^+ & \mapsto & 1, & w^- & \mapsto & 0, & x^+ & \mapsto & 0, & x^- & \mapsto & 1, \\ y^+ & \mapsto & 0, & y^- & \mapsto & 0, & z & \mapsto & 1 \end{array} \right\}$$

- $\mathcal{M}$ defines $(w \wedge \neg x)$, i.e., $(w \wedge \neg x) \models \exists z : \varphi$
  - Then add blocking clause and proceed
- Observe: $(w \wedge \neg x)$ under-approximates $\exists z : \varphi$
- So how about applying this to $\neg\varphi$?

Pushing Negations Around

$$\nu \models \forall z : \neg\varphi \quad \text{iff} \quad \neg\forall z : \neg\varphi \models \neg\nu$$

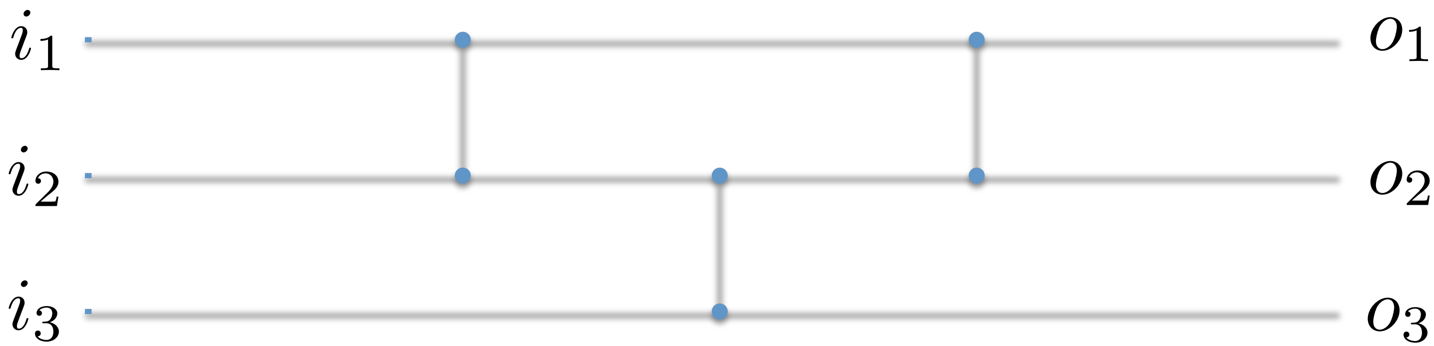
$$\text{iff} \quad \exists z : \varphi \models \neg\nu$$

- To find over-approximation $\neg\nu$ of $\exists z : \varphi$ compute under-approximation of $\forall z : \neg\varphi$
- But:
  - Can only derive implicants of $\exists z : \neg\varphi$
  - Not implicants of $\forall z : \neg\varphi$

Strategy for Over-Approximating Implicants

- Observe that $\forall z : \neg\varphi \models \exists z : \neg\varphi$
  - A model of $\forall z : \neg\varphi$ is also a model of $\exists z : \neg\varphi$
  - But not vice versa
- **Algorithm**:
  - Negate φ to obtain $\tau(\neg\varphi)$
  - Enumerate implicants C of $\exists z : \neg\varphi$
  - Filter those C such that $C \not\models \forall z : \neg\varphi$
  - Then $\exists z : \varphi \models \neg C$

Shortest Implicants: Sorting Networks



- Suppose sorter encoded as σ
- Cardinality constraint $i_1 + i_2 + i_3 = 2$ encoded as $o_1 \wedge o_2 \wedge \neg o_3$ in unary encoding
- $\tau(\neg\varphi) \wedge \sigma \wedge \bigwedge_{i=1}^{k} o_i \wedge \bigwedge_{i=k+1}^{n} \neg o_i$ specifies implicants of length k

Worked Example

- Take $\tau(\neg\varphi)$
- First, $\nu_1 = (\neg w)$ but $\exists z : \varphi \not\models \neg\nu_1$, so discard
- Then, $\nu_2 = (x)$ and $\exists z : \varphi \models \neg\nu_2$
- No more implicants of length 1
- Now, $\nu_3 = (\neg w \wedge \neg y)$ and $\exists z : \varphi \models \neg\nu_3$
- No more implicants, thus $\exists z : \varphi = (\neg x) \wedge (w \vee y)$

Some Experiments

- Written in Java on top of SAT4J
- Benchmark set from CNF encodings of ISCAS-85 hardware circuits
- Observed small CNF representation for quantifier-free formulae
- Runtime suffers from spurious candidates
  - Can be mitigated to some extent using co-factoring
- Traditional SAT-based algorithms rely on model enumeration (giving a DNF stored in BDDs)
  - If too expensive, no result can be computed
  - Our algorithm can still compute over-approximation

So as to not Cause Offense

- McMillan (CAV‘02)
- Lahiri et al. (CAV‘03 & CAV’06)
- Monniaux (CAV’10)
- Kettle et al. (TACAS‘06)
- Bryant (IEEE’87)
- Manquinho et al. (ICTAI‘97)
- Brauer et al. (CAV‘11)
- And many more ...

Conclusion

- Based on dual-rail encoding to derive implicants
- Combined with sorting networks so as to obtain shortest prime implicants
- Start with over-approximation which is then incrementally refined
- Algorithm is thus *anytime*

Thank you very much!